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De utriusque analyseos recentioris determinandi rationibus etc.

DE UTRISQUE ANALYSEOS RECENTIORIS DETERMINANDI RATIONIBUS ET EX UTRACUNQUE DETERMINATIONE IN ALTERAM TRANSITU LIBELLUS

AUCTORE

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D e u t r i s q u e
analyseos recentioris
determinandi rationibus,
e t
ex utracunque determinatione in alteram
t r a n s i t u.

Analysi, numerorum scientiae complexorum, ante inventam legum, datas res conjectandi scientiam sive artem combinatoriam fines perangustos praescriptos fuisse, notum. In nexandis numeris complexis atque relationibus illo e nexu exortis scientia illa versatur. Cui autem legibus opus est, data elementa conjectandi; artem combinatoriam oportet praecedere analysi.

Itaque *Hindenburgo*, geometrae celeberrimo, faciem eum scientiae nomine convenientem analyseos debemus; eum primum possibilitatem necessitudinemque accuratioris artis combinatoriae expositionis ante oculos posuisse, quisque in mathesi hospes non ignorat. Qui igitur geometra non solum illam scientiam, sed etiam excelsissimam matheseos partem, analysin, procreavit.

Quae ad merita maxime gravia et ad difficultatem generalem, integram rem ex nihilo procreandi, spectantes, Hindenburgum vitiorum, quamquam non tenuum, quibus laborat ejus expositio, non accusare debemus. Quae autem non in nulla integritate operationum deducendarum, sed multo magis in methodo versabatur et notatione. Quod hoc ultimum attinet, hic mentionem facere non audeo ejus, quod notatio combinatoria Hindenburgi nec sit generalis neque scientifica. Qua ex causa scientia conjectandi et methodus combinatoria in analysi non solum extra Germaniam, atque praesertim in Gallia, ubi *élegans analyticis* notatio illa nunquam placebit, sed etiam in patria ejus ipsa satis differri non potuit. Magis autem propagationem hujus scientiae methodus scholae hindenburgianae impediit, quae eam vocabat *artem combinatoriam*, sed etiam ut artem tractavit. Tironis erat, regulas, complexiones combinatorias procreandi, sine argumento accipere, sine demonstratione, insuper etiam in has, tali modo acceptas scientiam fundare, quam excelsissimam habebat partem totius matheseos. Scientia, datas res conjectandi, si pro fundamento habeatur analyseos, necesse est, mathesis elementaris non evidentius hac scientia exponatur ac apertius.

In analysi ut nanciscamur maximam generalitatem eo, quo alias casus tantum speciales considerari possunt, scientia efficit conjectandi. Qua enim omissa determinationes utraeque (independens et recurrens) in quibus vere analyseos natura versatur, nonnisi in perpaucis casibus specialibus dari possunt.

Ea, in qua versetur analysis, est *series* in genere. Nonnulli nexus directe sequentes quantitatum et omnino rerum secundum legem semper eandem si oriantur, hae singulae partes terminos constituunt seriei in genere. Functione alicujus quantitatis primariae consideranda si pro ea primaria valores determinati sequentes, successive substituantur, series existit, pro qua functio illa expressio formalis est legis procreandi. Classes successivae per aliquam operationem combinatoriam orientes secundum legem effluunt eandem, atque igitur terminos constituunt seriei.

Qui autem termini eadem lege procreati, inter se quoque nexum continent certum; semper exsistit regula, per quam ex nonnullis terminis prioribus sequentes deduci possunt. Quisque terminus per se, et tantum ex quantitatibus vel rebus, quarum e nexu terminos illos oriuntur, independenter a quocunque alio produci potest, quae est *solutio independens problematis*; attamen etiam ex terminis seriei prioribus jam productis sequens gigni potest, quae est *determinatio recurrens.* Terminus generalis seriei regulam independentem formalem constituit, regula recurrens semper significatione analytica facile exprimi potest, atque tum *formula recursionis* vocatur.

Postulato terminorum vere computo determinatio independens gravior est recurrente quamquam saepe commodiori, scientiae autem ipsi ambae pari utilitate, pari sunt necessitudine. Persaepe enim series operatione aliqua jam procreatae denuo ad novas nexantur relationes, quo praesertim termini generales sunt considerandi, ut iis agantur ea, quae seriebus ipsis facere decernebantur Quo faciendo autem necesse est, terminos illos generales independenter produci ab omnibus reliquis, quare determinationem hanc esse necessariam satis jam apparet. Pari autem modo determinatione independente analysi opus est. Plerumque enim disquisitionibus analyticis in determinationes inciditur recurrentes, atque ad independentem nonnisi per formulas recursionis aliarum quantitatum jam notas transire possumus.

Materia analyseos, ut supra jam diximus, est series, terminos earum inveniendi, ejus negotium. Quam determinationem duabus viis supra dictis praestari oportet. Dependit autem a casu singulari, an consideratione primitiva primum in determinationem independentem incidatur, an perveniatur ad recurrentem; ab altera in alteram transire, super est praetea. In nonnullis casibus perfacile ad formulam recursionis pervenitur, cujus autem usus in latiores inducit computos, quare statim determinationem independentem oportet deduci. Notum est, e. gr. inter classes combinatorias omissis repetitionibus sequentem exsistere relationem:

$$\overset{k}{C'}[1\,..\,n] = \overset{k-1}{C'}[1\,..(n-1)]\,.\,n + \overset{k}{C'}[1\,..(n-1)]$$

quaeratur numerus formarum combinatoriarum ipsius $\overset{k}{C'}[1\,..\,n]$.

Quem numerum, si per $S\overset{k}{C'}[1\,..\,n]$ indicetur, inter hos numeros relatio recurrens invenitur sequens:

$$S\overset{k}{C'}[1\,..\,n] = S\overset{k-1}{C'}[1\,..(n-1)] + S\overset{k}{C'}[1\,..(n-1)]$$

seu, si pro $\overset{k}{S}C'[1..n]$ signo utamur simpliciori, adest:

$$ {}^{n}\overset{k}{A} = {}^{n-1}\overset{k-1}{A} + {}^{n-1}\overset{k}{A} $$

recursionis formula generalis ad computandos numeros formarum combinatoriarum omissis repetitionibus. Cujus autem ope si computandus sit e. gr. $\overset{10}{S}C'[1..20]$, eum computum perdifficilem esse, et maxime latum, statim perspicitur; si autem nota sit solutio problematis independens:

$$ \overset{k}{S}C'[1..n] = {}^{n}\overset{k}{\mathfrak{B}} $$

perfacile solutio in:

$$ \frac{20.\,19.\ldots 11}{1.\,2.\ldots 10} = 184756 $$

habetur.

Hic igitur interrogabitur, quomodo ex formula recursionis:

$$ {}^{n}\overset{k}{A} = {}^{n-1}\overset{k-1}{A} + {}^{n-1}\overset{k}{A} $$

relatio independens:

$$ {}^{n}\overset{k}{A} = \overset{k}{S}C'[1..n] = {}^{n}\overset{k}{\mathfrak{B}} $$

deduci possit, i. e. generaliter, quomodo transitus ex recurrente determinatione ad independentem effici possit.

Exempla transitus ex independente solutione ad recurrentem mathematicorum scriptoribus veteribus nonnulla tantum reperiuntur, dum methodus a recursionis formula determinationem independentem deducendi modo non generali sub nomine inductionis passim in scriptis veterum versatur.

Via et ratio, ex altera determinatione in alteram transeundi, argumentum sit harum disquisitionum.

I.

De ex recurrente determinatione in alteram transitu, seu de inductione.

Inductionis ratio geometrae praesertim ante ejus amplificationem *) ad eam demonstrationem, quam dicunt methodum ab n ad n + 1, non generalem habuerunt et consilium semper dederunt, ne illa applicetur nisi summa necessitate cogente. Attamen demonstratio amplificata, methodus ab n ad n + 1, nostra aetate ipsa vituperatur, et censetur, illam, quamvis non ratione veritatis rei demonstrandae, sed saltem quod rationem et modum attineat, quomodo ad rem intelligendam perveniatur, non satis esse perspicuam.

*) Primum a Jac. Bernoulli applicata. Acta erud. 1686. pag. 360. Opp. T. I. nr. 24. Ars conject. pag. 93 seqq.

Demonstratio ope inductionis, si generaliter ducatur, nihil est aliud, nisi transitus ex recursionis determinatione ad independentem solutionem, atque ulla ratione pro evidente ac strenua haberi potest.

Expressio formalis legis, secundum quam termini alicujus seriei procreantur, sive terminus generalis hujus seriei ex una quantitate primaria constare potest, vel pluribus. Qui igitur termini ratione variarum quantitatum primariarum varie recurrere poterunt. Expressio ex duabus quantitatibus primariis, k et n, constans, tam ratione quantitatis k, quam n, seu respectu ambarum k et n recurrere potest.

Quantitates non easdem modo recurrere posse non eodem, atque quantitates eodem modo recurrentes esse quantitates easdem, satis perspicuum est.

Disquisitione primitiva si reperiatur, eam quantitatem, pro qua invenienda est lex independens, sequente modo recurrere:

$$ {}^{n}\overset{k}{A} = {}^{n-1}\overset{k-1}{A} + {}^{n-1}\overset{k}{A} $$

concludi potest, quantitatem ${}^{n}\overset{k}{A}$ esse coefficientem k^{tum} binomialem potentiae n^{tae} tali modo recurrentem:

$$ {}_{n}\overset{k}{\mathfrak{B}} = {}_{n-1}\overset{k-1}{\mathfrak{B}} + {}_{n-1}\overset{k}{\mathfrak{B}} . $$

Recursionis formulis congruentibus primum necesse est, eas eundem numerum habere tam terminorum quam quantitatum primariarum, deinde autem, terminos aequealtos ex iisdem quantitatibus primariis eodem modo esse constructos

In ambabus formulis recurrentibus ut cognoscatur earum identitas, in quantitates spectandum est nonnisi primarias, quantitatibus in recursione eaedem permanentibus et ad identitatem cognoscendam non pertinentibus non spectatis.

Congruunt e. gr. formulae recursionis sequentes:

$$ {}^{n}\overset{k}{A} = {}^{n-1}\overset{k}{A} . n + {}^{n}\overset{k-1}{A} $$

et

$$ \overset{n}{B}{}^{k} = \overset{n-1}{B}{}^{k} . n + \overset{n}{B}{}^{k-1} $$

attamen ambae cum formula

$$ {}^{n+m}\overset{k+h}{F} = {}^{n+m-1}\overset{k+h}{F}{}_{(n+m)} + {}^{n+m}\overset{k+h-1}{F} $$

quoque congruunt, ubi m et h sunt quantitates constantes, termini enim hujus formulae eodem modo ex quantitatibus primariis k, n sunt compositi, ut ii praecedentium formularum.

Sequitur ergo, identitate duarum recursionum conclusa ad quamque quantitatem primariam formulae recursionis repertae insuper constantem aliquam addi oportere, ut disquisitio sit generalis.

Quam constantem in quocunque casu singulari adhuc esse determinandam, satis elucet, ad quod efficiendum methodo, ei determinationis constantis in calculo integrali simili, opus est.

In permultis analyseos disquisitionibus observatione ad aliquam pervenitur legem inductionis ope tum demonstrandam. Methodo hac generali in inductionis demonstrationibus sive conclusione identitatis duarum recursionum, seu, quod idem, transitu ex recurrente determinatione ad independentem, ducti, quaesitum sine magno negotio et, quod rei summum, generaliter nanciscimur.

Quare haec probandi ratio in analysi magna est utilitate atque necessitudine. Quae autem ut applicari possit, determinationes recurrentes seu formulas recursionis variarum quantitatum gravium esse cognitas necesse est.

Sunt nümeri, in quos persaepe in disquisitionibus analyticis incidatur, qui *facultates (factorielles)* vocantur.

Expressio. ${}^{n}\overset{k}{\mathfrak{F}}{}^{d}$ si indicet facultatem baseos n, exponentis k et differentiae d, i. e. si

$$ {}^{n}\overset{k}{\mathfrak{F}}{}^{d} = n(n+d)(n+2d)\dots(n+kd) $$

adest inter hos numeros recursio sequens:

$$ {}^{n}\overset{k}{\mathfrak{F}}{}^{d} = (k+1)d.\,{}^{n}\overset{k-1}{\mathfrak{F}}{}^{d} + {}^{n-d}\overset{k}{\mathfrak{F}}{}^{d} $$

pro d = 1 invenitur:

$$ {}^{n}\overset{k}{\mathfrak{F}}{}^{1} = (k+1)\,{}^{n}\overset{k-1}{\mathfrak{F}}{}^{1} + {}^{n-1}\overset{k}{\mathfrak{F}}{}^{1} $$

pro d = — 1 accipitur denique:

$$ {}^{n}\overset{k}{\mathfrak{F}}{}^{-1} = -(k+1)\,{}^{n}\overset{k-1}{\mathfrak{F}}{}^{-1} + {}^{n+1}\overset{k}{\mathfrak{F}}{}^{-1} \text{ sive} $$

$$ {}^{n}\overset{k}{\mathfrak{F}}{}^{-1} = {}^{n-1}\overset{k}{\mathfrak{F}}{}^{-1} + {}^{n-1}_{k+1}\overset{k-1}{\mathfrak{F}}{}^{-1}. $$

Discerptione formulae specialis ex generali eo exortae, ut pro d valor substuatur — 1 sequentes formulae totales deduci possunt:

$$ \frac{1}{k+1}.\,{}^{n}\overset{k}{\mathfrak{F}}{}^{1} = {}^{n}\overset{k-1}{\mathfrak{F}}{}^{1} + {}^{n-1}\overset{k-1}{\mathfrak{F}}{}^{1}\dots + {}^{n-h}\overset{k-1}{\mathfrak{F}}{}^{1}\dots\ {}^{1}\overset{k-1}{\mathfrak{F}}{}^{1} $$

et

$$ {}^{n}\overset{k}{\mathfrak{F}}{}^{1} = {}^{n-1}\overset{k}{\mathfrak{F}}{}^{1} + {}^{k+1}\overset{0}{\mathfrak{F}}{}^{-1}.\,{}^{n-1}\overset{k-1}{\mathfrak{F}}{}^{1}\dots + {}^{k+1}\overset{h-1}{\mathfrak{F}}{}^{-1}. $$

$$ {}^{n-1}\overset{k-h}{\mathfrak{F}}{}^{1}\dots + {}^{k+1}\overset{k-1}{\mathfrak{F}}{}^{-1}\,{}^{n-1}\overset{0}{\mathfrak{F}}{}^{1} + {}^{k+1}\overset{0}{\mathfrak{F}}{}^{-1} $$

Discerptione alterius formulae ex generali substitutione — 1 pro d deductae, sequentes totales inveniuntur:

$$\frac{1}{k+1} \cdot {}^{n}\overset{k}{\mathfrak{F}}{}^{-1} = {}^{n-1}\overset{k-1}{\mathfrak{F}}{}^{-1} + {}^{n-2}\overset{k-1}{\mathfrak{F}}{}^{-1}_{\dots} + {}^{n-h}\overset{k-1}{\mathfrak{F}}{}^{-1}_{\dots} + {}^{k}\overset{k-1}{\mathfrak{F}}{}^{-1}$$

atque

$$\begin{aligned} {}^{n}\overset{k}{\mathfrak{F}}{}^{-1} = {}^{n-1}\overset{k}{\mathfrak{F}}{}^{-1} &+ {}^{k+1}\overset{0}{\mathfrak{F}}{}^{-1} \cdot {}^{n-2}\overset{k-1}{\mathfrak{F}}{}^{-1}_{\dots} + {}^{k+1}\overset{h-1}{\mathfrak{F}}{}^{-1} \cdot \\ {}^{n-(h+1)}\overset{k-h}{\mathfrak{F}}{}^{-1}_{\dots} &+ {}^{k+1}\overset{k-1}{\mathfrak{F}}{}^{-1}\; {}^{n-(k+1)}\overset{0}{\mathfrak{F}}{}^{-1} + {}^{k+1}\overset{k}{\mathfrak{F}}{}^{-1} \end{aligned}$$

Quae recursionis formulae sex in transitibus saepissime versantur.

Relationes quoque recurrentes potentiarum expontium integrorum non sunt absque omni applicatione. Quarum prima adest:

$$a^n = a.\ a^{n-1}$$

qua transmutata sit:

$$a^n = (a-1)\, a^{2-n} + a^{n-1}$$

qua ex partiali sequens deducitur totalis:

$$a^n = (a-1)\left[a^{n-1} + a^{n-2}_{\dots} + a^{n-h}_{\dots}\, a^0\right] + 1$$

et pro $a = 2$ magis simplificata:

$$2^n = 2^{n-1} + 2^{n-2}_{\dots} + 2^{n-h}_{\dots}\, 2^0 + 1$$

Expressiones combinatorias praesertim in disquisitionibus versari analyticis, quisque non ignorabit. Formulas igitur recursionis combinatorias graviores notas oportet.

Combinationes omissis repetitionibus elementorum ad sequentem traducunt recursionis formulam partialem:

$$\overset{k}{C'}[1..n] = 1.\overset{k-1}{C'}[2..n] + \overset{k}{C'}[2..n]$$

qua discerpta sequentes producuntur formulae recursionis totales:

$$\overset{k}{C'}[1..n] = 1.\overset{k-1}{C'}[2..n] + 2.\overset{k-1}{C'}[3..n] \ldots$$

$$+ r.\overset{k-1}{C'}[(r+1)..n]..+(n-k+1).\overset{k-1}{C'}[(n-k+2)..n]$$

et:

$$\overset{k}{C'}[1..n] = \overset{k}{C'}[2..n] + 1\overset{k-1}{C'}[3..n] \ldots$$

$$+ 1.2...r.\overset{k-r}{C'}[(r+2)..n]\ldots + 1..k.\overset{0}{C'}[(k+2)..n]$$

Transformatione illius formulae recursionis partialis confecta emergit:

$$\overset{k}{C'}[1..n] = \overset{k}{C'}[1..(n-1)] + \overset{k-1}{C'}[1..(n-1)].n$$

cujus discerptione inveniuntur duae recursionis formulae totales:

$$\overset{k}{C'}[1..n] = \overset{k-1}{C'}[1..(n-1)].n + \overset{k-1}{C'}[1..(n-2)](n-1)\ldots$$

$$+ \overset{k-1}{C'}[1..(n-r)](n-(r-1))\ldots + \overset{k-1}{C'}[1..(k-1)].k$$

atque:

$$\overset{k}{C'}[1..n] = \overset{k}{C'}[1..(n-1)] + \overset{k-1}{C'}[1..(n-2)].n\ldots$$

$$+ \overset{k-r}{C'}[1..(n-(r+1))](n-(r-1))..n + \overset{0}{C'}.(n-(k-1)..n,$$

Docebit autem transmutatio perfacilis hujus partialis, esse tam:

$$\overset{k}{C}`[1..n] = \overset{k}{C}`[1..(n+1)] - \overset{k-1}{C}`[1..n](n+1)$$

quam:

$$\overset{k}{C}`[1..n] = \frac{\overset{k+1}{C}`[1..(n+1)] - \overset{k+1}{C}`[1..n]}{n+1}$$

ex quibus emergunt quatuor totales sequentes:

$$\overset{k}{C}`[1..n] = \overset{k}{C}`[1..(n+1)] - \overset{k-1}{C}`[1..(n+1)](n+1) \ldots + (-1)^h . \overset{k-h}{C}`[1..(n+1)](n-1)^h \ldots + (-1)^{k-1} . \overset{1}{C}`[1..(n+1)](n+1)^{k-1} + (-1)^k (n+1)^k$$

$$\overset{k}{C}`[1..n] = \overset{k}{C}`[1..(n+r)] - \overset{k-1}{C}`[1..n](n+1) - \ldots - \overset{k-1}{C}`[1..(n+h-1)](n+h) \ldots - \overset{k-1}{C}`[1..(n+r-1)].(n+r)$$

ubi r est numerus arbitrarius.

$$\overset{k}{C}'[1..n] = \frac{\overset{k+1}{C}'[1..(n+1)](n+1)^{n-k} - \ldots + (-1)^{h-1}\,\overset{k+h}{C}'[1..(n+1)](n+1)^{n-k-(h-1)} \ldots + (-1)^{n+k}\,\overset{n+1}{C}'[1..(n+1)](n+1)^{0}}{(n+1)^{n-(k-1)}}$$

$$\overset{k}{C}'[1..n] = \frac{\overset{k+r}{C}'[1..(n+r)] - \overset{k+1}{C}'[1..n](n+r)\ldots(n+2) - \ldots - \overset{k+h}{C}'[1..(n+h-1)](n+r)\ldots(n+h+1)\ldots - \overset{k+r}{C}'(1..(n+r-1)]}{(n+r+1).(n+r)\ \ldots\ (n+1)}$$

ubi iterum r est numerus arbitrarius.

Combinationibus admissis repetitionibus considerandis recursionis nanciscimur formulam:

$$\overset{k}{C}[1..n] = 1.\overset{k-1}{C}[1..n] + \overset{k}{C}[1..n]$$

qua discerpta habetur:

$$\overset{k}{C}[1..n] = 1.\overset{k-1}{C}[1..2] + 2.\overset{k-1}{C}[2..n] \ldots + r.\overset{k-1}{C}[r..n] \ldots + n\overset{k-1}{C}[n]$$

et:

$$\overset{k}{C}[1..n] = \overset{k}{C}[2..n] + 1.\overset{k-1}{C}[2..n] \ldots + 1^{r}\overset{k-r}{C}[2..n] \ldots + 1^{k}.\overset{0}{C}[2..n]$$

Quae formula autem si transmutetur in relationem:

$$\overset{k}{C}[1..n] = \overset{k-1}{C}[1..n].n + \overset{k}{C}[1..(n-1)]$$

transit.

Discerptione conferta evenit:

$$\overset{k}{C}[1..n] = \overset{k-1}{C}[1..n].n + \overset{k-1}{C}[1..(n-1)](n-1) \ldots$$
$$+ \overset{k-1}{C}[1..(n-r)](n-r) \ldots + \overset{k-1}{C}[1].$$

et:

$$\overset{k}{C}[1..n] = \overset{k}{C}[1..(n-1)] + \overset{k-1}{C}[1..(n-1)]n \ldots$$
$$+ \overset{k-r}{C}[1..(n-1)].n^r \ldots + \overset{o}{C}[1..(n-1).n^k$$

Sequitur insuper ex illa recursionis formula partiali, esse primum:

$$\overset{k}{C}[1..n] = \overset{k}{C}[1..(n+1)] - \overset{k-1}{C}[1..(n+1)](n+1)$$

deinde autem:

$$\overset{k}{C}[1..n] = \frac{\overset{k}{C}[1..n] - \overset{k+1}{C}[1..(n-1)]}{n}$$

Quibus discerptis reperiuntur quatuor sequentes:

$$\overset{k}{C}[1..n] = \overset{k}{C}[1..(n+1)] - \ldots + (-1)^h.$$
$$\overset{k-h}{C}[1..(n+h-1)](n+1)..(n+h) \ldots + (-1)^k$$
$$\overset{o}{C}[1..(n+k)](n+1)..(n+k)$$

$$\overset{k}{C}[1..n] = \overset{k}{C}[1..(n+r)] - \overset{k-1}{C}[1..(n+r)](n+r)\ldots - \overset{k-1}{C}[1..(n+r-h)](n+r-h)\ldots$$
$$- \overset{k-1}{C}[1..(n+1)](n+1)$$

ubi r numerus est arbitrarius.

$$\overset{k}{C}[1..n] = \frac{\overset{k+1}{C}[1..n]\,1.2..(n+1) - \ldots + (-1)^{h-1}\overset{k+h}{C}[1..(n-(h-1))]\,1..(n-h)\ldots + (-1)^{n-1}\overset{k+n}{C}[1]}{n.(n-1).(n-2)\ \ldots\ 2.1}$$

$$\overset{k}{C}[1..n] = \frac{\overset{k+r}{C}[1..n] - \overset{k+r}{C}[1..(n-1)]\ldots - \overset{k+r-h}{C}[1..(n-1)]\,n^{h}\ \ldots - \overset{k+1}{C}[1..(n-1)]\,n^{r-1}}{n^{r}}$$

Combinationes autem numeri propositi elementis autem non repetendis inter se sequentem habent relationem recurrentem:

$${}^{n}\overset{k}{C}'[q..] = {}^{n-q}_{q.}\overset{k-1}{C}'[(q+1)..] + {}^{n}\overset{k}{C}'[(q+1)..]$$

ubi q numerum repraesentat tam positivum quam negativum. Pro $q = 1$ adest:

$${}^{n}\overset{k}{C}'[1..] = {}^{n-1}_{1.}\overset{k-1}{C}'[2..] + {}^{n}\overset{k}{C}'[2..]$$

Discerptione generalioris formulae facta accipitur:

$$ {}^{n}\overset{k}{C}{}'[q..] = {}^{n-q}_{q.}\overset{k-1}{C}{}'[(q+1)..] + {}^{n-(q+1)}_{(q+1).}\overset{k-1}{C}{}'[(q+2)..]\ldots $$

$$ + {}^{n-(q+h)}_{(q+h).}\overset{k-1}{C}{}'[(q+h+1)..]. $$

ubi autem h non majorem quantitate $\frac{n}{k} - \frac{k-1}{2}$ — q esse oportet.

Pro $q = 1$ adest:

$$ {}^{n}\overset{k}{C}{}'[1..] = {}^{n-1}_{1.}\overset{2-1}{C}{}'[2..] + {}^{n-2}_{2.}\overset{k-1}{C}{}'[3..]\ldots $$

$$ + {}^{n-h}_{h}\overset{k-1}{C}{}'[(h+1)..] $$

ubi iterum h non major quam $\frac{n}{k} - \frac{k-1}{2}$ esse potest.

Combinationes numeri propositi admissis repetitionibus in sequentem inducunt relationem:

$$ {}^{n}\overset{k}{C}[q..] = {}^{n-q}_{q.}\overset{k-1}{C}[(q+1).] + {}^{n}\overset{k}{C}[(q+1)..] $$

pro $q = 1$ adest:

$$ {}^{n}\overset{k}{C}[1..] = {}^{n-1}_{1.}\overset{k-1}{C}[1..] + {}^{n}\overset{k}{C}[2..]. $$

Discerptio hac formula indicata si conficiatur, emergit:

$$ {}^{n}\overset{k}{C}[q..] = {}^{n-q}_{q.}\overset{k-1}{C}[q..] + {}^{n-(q+1)}_{(q+1).}\overset{k-1}{C}[(q+1)..]\ldots $$

$$ + {}^{n-(q+r)}_{(q+r)}\overset{k-1}{C}[(q+r)..] $$

ubi autem r non majorem esse, quam $\frac{n}{k}$ — q, necesse est.

Pro $q = 1$ invenitur:

$$ {}^{n}\overset{k}{C}[1..] = {}^{n-1}_{1.}\overset{k-1}{C}[1..] + {}^{n-2}_{2.}\overset{k-1}{C}[1..] \ldots + {}^{n-r}_{r.}\overset{k-1}{C}[r..] $$

in qua r non major est quam $\frac{n}{k}$.

Procreatur insuper ex priori formula recursionis partiali totalis sequens:

$$ {}^{n}\overset{k}{C}[q..] = {}^{n}\overset{k}{C}[(q+1)..] + {}^{n-q}_{q.}\overset{k-1}{C}[(q+1)..] \ldots $$
$$ + {}^{n-hq}_{q.}{}^{h}\overset{k-h}{C}[(q+1)..] \ldots + {}^{n-(k-1)q}_{q}{}^{k-1}_{.}\overset{1}{C}[(q+1)..] $$

ubi casu accepto, in quo $k(q+1) - n$, is terminus est realis primus, cui est exponens classis $k - [k(k+1) - n]$. Pro $q = 1$ adest:

$$ {}^{n}\overset{k}{C}[1..] = {}^{n}\overset{k}{C}[2..] + {}^{n-1}_{1}\overset{k-1}{C}[2..] \ldots + {}^{n-h}_{1.}{}^{h}\overset{k-h}{C}[2..] \ldots $$
$$ {}^{n-(k-1)}_{1}{}^{k-1}_{.}\overset{1}{C}[2..] $$

in qua formula eadem conditione paria sunt respicienda.

Legibus determinationis recurrentis combinatoriarum classium, quas summa proposita admittit, deducendis, in formulam inciditur sequentem:

$$ {}^{n}C[1..] = {}^{n-1}_{1.}C[1..] + {}^{n}C[2..] $$

qua discerpta exsistit pro n numero posito pari:

$$ {}^{2n}C[1..] = {}^{2n-1}_{1.}C[1..] + {}^{2n-2}_{2.}C[2..] \ldots + $$
$$ {}^{2n-r}_{r.}C[r..] .. + {}^{n}_{n}C[n..] + 2n. $$

inpari autem:

$$ {}^{2n+1}C[1..] = {}^{2n}_{1.}C[1..] + {}^{2n-1}_{2.}C[2..] \ldots $$
$$ + {}^{2n-(r-1)}_{r.}C[r..] \ldots + {}^{n+1}_{n.}C[n..] + 2n + 1 $$

Variationibus investigandis reperitur formula recurrens:

$$ \overset{k}{V}[1..n] = 1.\overset{k-1}{V}[1..n] + 2.\overset{k-1}{V}[1..n] \ldots + h.\overset{k-1}{V}[1..n] \ldots $$
$$ + n.\overset{k-1}{V}[1..n] $$

sive

$$ \overset{k}{V}[1..n] = \overset{k-1}{V}[1..n].1 + \overset{k-1}{V}[1..n]2.. + \overset{k-1}{V}[1..n]h.. $$
$$ + \overset{k-1}{V}[1..n].n $$

Quae autem formula, seriebus elementorum positis aequalibus, si per ${}_{p}\overset{k}{C}[1..n]$ indicetur, quamque formam ipsius $\overset{k}{C}[1..n]$ multiplicari numero permutationis ejus, in sequentes transmutantur:

$$ {}_{p}\overset{k}{C}[1..n] = 1.{}_{p}\overset{k-1}{C}[1..n] + 2.{}_{p}\overset{k-1}{C}[1..n] \ldots $$
$$ + h.{}_{p}\overset{k-1}{C}[1..n] \ldots + n.{}_{p}\overset{k-n}{C}[1..n] $$

seu

$$\overset{k}{{}_pC}[1..n] = \overset{k-1}{{}_pC}[1..n].1 + \overset{k-1}{{}_pC}[1..n].2 \ldots$$
$$+ \overset{k-1}{{}_pC}[1..n].h \ldots + \overset{k-1}{{}_pC}[1..n].n$$

In variationum numeri propositi theoria deducuntur formulae recursionis:

$$\overset{k}{{}^nV}[q..] = {}^{n-q}_{q.}\overset{k-1}{V}[q..] + {}^{n-(q+1)}_{(q+1).}\overset{k-1}{V}[q..] \ldots$$
$$+ {}^{n-(q+r)}_{(q+r).}\overset{k-1}{V}[q..] \ldots + {}^{(k-1)q}_{n-(k-1)q}\overset{k-1}{V}[q..]$$

sive

$$\overset{k}{{}^nV}[q..] = {}^{n-q}\overset{k-1}{V}[q..]q + {}^{n-(q+1)}\overset{k-1}{V}[q..](q+1) \ldots$$
$$+ {}^{n-(q+r)}\overset{k-1}{V}[q..](q+r) \ldots + {}^{(k-1)q}\overset{k-1}{V}[q..](n-(k-1)q)$$

in quibus q elementum indicis positivi vel negativi esse potest. Pro $q = 1$ sequentes transeunt:

$$\overset{k}{{}^nV}[1..] = {}^{n-1}_{1.}\overset{k-1}{V}[1..] + {}^{n-2}_{2.}\overset{k-1}{V}[1..] \ldots$$
$$+ {}^{n-r}_{r.}\overset{k-1}{V}[1..] \ldots + {}^{k-1}_{n-(k-1)}\overset{k-1}{V}[1..]$$

sive:

$$\overset{k}{{}^nV}[1..] = {}^{n-1}\overset{k-1}{V}[1..].1 + {}^{n-2}\overset{k-1}{V}[1..].2 \ldots$$
$$+ {}^{n-r}\overset{k-1}{V}[1..].r \ldots + {}^{k-1}\overset{k-1}{V}[1..].(n-(k-1))$$

Elementorum autem seriebus acceptis aequalibus eaedem formulae recursionis pro $\overset{k}{{}^nV}[q..]$

et ${}^{n}\overset{k}{V}[1..]$ exoriuntur, sic autem, ut pro ${}^{n}\overset{k}{V}[q..]$ substituatur ${}^{n}_{p}\overset{k}{C}[q..]$.

Classibus denique variationis omnibus in eandem summam spectantibus deducendis recursio derivatur sequens:

$$
\begin{aligned}
{}^{n}V[1..] &= {}^{n-1}_{1.}V[1..] + {}^{n-2}_{2.}V[1..] \ldots \\
&\quad + {}^{n-r}_{r.}V[1..] \ldots {}^{1}_{(n-1)}V[1..] + n
\end{aligned}
$$

seu elementorum seriebus iisdem:

$$
\begin{aligned}
{}^{n}_{p}C[1..] &= {}^{n-1}_{1.p}C[1..] + {}^{n-2}_{2.p}C[1..] \ldots \\
&\quad + {}^{n-r}_{r.p}C[1..] \ldots + {}^{1}_{(n-1)p}C[1..] + n
\end{aligned}
$$

Relationibus igitur gravium quantitatum et expressionum combinatoriarum recurrentibus notis, non est difficile, probandi ratione uti supra dicta.

Primum igitur omnium formarum operationibus combinatoriis diversis deducendarum numerum determinabimus, exempla transitus ex recurrente solutione in independentem ut praebeamus.

Signo $P[1..n]$ si utamur, ut omnes indicentur formae permutationis, quae ex elementis n vere diversis procreari possunt, signum $SP[1..n]$ numerum illarum formarum exhibeat. Ita etiam ratione signorum $S\overset{k}{C}[1..n]$, $S\overset{k}{V}[1..n]$ ctr.

Inter classes permutationum sequens habetur relatio recurrens:

$$P[..n] = 1 . P\left[\frac{1..n}{1}\right] + 2 . P\left[\frac{1..n}{2}\right] \dots + r . P\left[\frac{1..n}{r}\right] \dots + n . P\left[\frac{1..n}{n}\right]$$

adest ergo:

$$SP[1..n] = SP\left[\frac{1..n}{1}\right] \dots + SP\left[\frac{1..n}{r}\right] \dots + SP\left[\frac{1..n}{n}\right].$$

relatio recurrens inter numeros quaesitos. Cujus autem terminus quisque, quorum omnino exstant n, numerum indicat formarum permutationum ex n - 1 elementis vere diversis confectarum; est igitur:

$$SP[1..n] = n . SP[1..(n-1)]$$

Qui numeri (numeri permutationis) ab una tantum quantitate primaria dependunt, signis itaque pro iis positis simplicioribus, regulam recursionis eorum ita exprimere possumus:

$$\overset{n}{A} = n . \overset{n-1}{A}$$

Attamen tali modo quoque ${}_1\overset{n}{\mathfrak{F}}{}_1$ recurrit, inter quos enim sequens exsistit recursio:

$${}_1\overset{n}{\mathfrak{F}}{}_1 = n . {}_1\overset{n-1}{\mathfrak{F}}{}_1$$

est igitur constante rite addita:

$$SP[1..n] = {}^{1}\overset{n+h}{\mathfrak{F}}{}^{1}$$

Ad constantem determinandam monemus, pro $n = 1$ ut sit $SP[1..n] = 1$, ex quo colligitur, esse $h = -1$ atque igitur:

$$SP[1..n] = {}^{1}\overset{n-1}{\mathfrak{F}}{}^{1} = 1.2.3 \ldots n$$

Ex recursionis formula supra posita combinationum omissis repetitionibus sequitur, esse:

$$S\overset{k}{C}'[1..n] = S\overset{k-1}{C}'[1..(n-1)] + S\overset{k}{C}'[1..(n-1)]$$

sive substitutis signis simplicioribus:

$${}^{n}\overset{k}{A} = {}^{n-1}\overset{k-1}{A} + {}^{n-1}\overset{k}{A}$$

eodem autem modo quoque formula:

$${}^{n}\overset{k}{\mathfrak{F}}{}^{-1} = {}^{n-1}\overset{k-1}{\mathfrak{F}}{}^{-1} . (k+1) + {}^{n-1}\overset{k}{\mathfrak{F}}{}^{-1}$$

recurreret, nisi multiplicaretur terminus ${}^{n-1}\overset{k-1}{\mathfrak{F}}{}^{-1}$ factore $(k+1)$. Quae tamen formula ita transformari potest, ut sit illi aequalis. Ad quod efficiendum quemque terminum oportet multiplicari numero $p^{-(k+1)}$, seu dividi per $p^{\frac{1}{k+1}}$, quo facto exsistit:

$$p^{-(k+1)} . {}^{n}\overset{k}{\mathfrak{F}}{}^{-1} = p^{-k} . {}^{n-1}\overset{k-1}{\mathfrak{F}}{}^{-1} + p^{-(k+1)} . {}^{n-1}\overset{k}{\mathfrak{F}}{}^{-1}$$

qua ex identitate sequitur esse constantibus rite fictis:

$$S\overset{k}{C'}[1..n] = p^{-(k+h+1)} \cdot {}^{n+g}\overset{k+h}{\mathfrak{F}}{}^{-1}$$

ad quas inveniendas monemus, ut sit pro $k = 1$, $S\overset{k}{C'}[1..n] = n$, ex quo colligi potest, esse $h = -1$ atque $g = 0$, ita, ut reperiatur:

$$S\overset{k}{C'}[1..n] = p^{-k} \cdot {}^{n}\overset{k-1}{\mathfrak{F}}{}^{-1} = \frac{n(n-1)\ldots(n-(k-1))}{k,(k-1)\ldots 2.1}$$

Eodem fere modo numerus formarum combinatoriarum admissis repetitionibus deduci potest.

Inter classes combinationum admissis repetitionibus exstat relatio recurrens:

$$\overset{k}{C}[1..n] = \overset{k-1}{C}[1..n].n + \overset{k}{C}[1..(n-1)]$$

ex qua cognoscitur, esse:

$$S\overset{k}{C}[1..n] = S\overset{k-1}{C}[1..n] + S\overset{k}{C}[1..(n-1)]$$

sive signis acceptis simplicioribus:

$${}^{n}\overset{k}{A} = {}^{n}\overset{k-1}{A} + {}^{n-1}\overset{k}{A}$$

qua cum recursionis formula consentiret:

$${}^{n}\overset{k}{\mathfrak{F}}{}^{1} = {}^{n}\overset{k-1}{\mathfrak{F}}{}^{1}.(k+1) + {}^{n-1}\overset{k}{\mathfrak{F}}{}^{1}$$

nisi in termino illo factor $k+1$ inesset. Quibus autem terminis numero $p^{-(k+1)}$ multiplicatis exoritur.

$$p^{-(k+1)}\, {}^{n}\overset{k}{\mathfrak{F}}{}^{1} = p^{-k} \cdot {}^{n}\overset{k-1}{\mathfrak{F}}{}^{1} + p^{-(k+1)} \cdot {}^{n-1}\overset{k}{\mathfrak{F}}{}^{1}$$

quarum recursionum identitate sequitur, esse:

$$S\overset{k}{C}[1..n] = {}^{-(k+h+1)}_{\quad p}\;.\;{}^{n+g}\overset{k+h}{\mathfrak{F}}{}^{1}$$

pro $k = 1$, $S\overset{k}{C}[1..n] = n$ invenitur, ex quo deduci potest, esse $h = -1$ atque $g = 0$, quibus valoribus pro h et g positis emergit:

$$S\overset{k}{C}[1..n] = {}^{k}_{p}.\;{}^{n}\overset{k-1}{\mathfrak{F}}{}^{1} = \frac{n(n+1)\ldots(n+(k-1))}{k.(k-1)\ldots 2.1}$$

Variationes in sequentem inducunt relationem:

$$\overset{k}{V}[1..n] = \overset{k-1}{V}[1..n]. + \overset{k-1}{V}[1..n].2 \ldots$$

$$+ \overset{k-1}{V}[1..n].h \ldots + \overset{k-1}{V}[1..n].n$$

cujus termini, quorum sunt n omnes numerum eundem habent formarum, quare habetur:

$$S\overset{k}{V}[1..n] = n.S.\overset{k-1}{V}[1..n]\,;$$

seu si signis utamur simplicioribus:

$${}^{n}\overset{k}{A} = n.{}^{n}\overset{k-1}{A}$$

Eodem autem modo recurrent potentiae ipsius n, nam invenitur:

$$n^{k} = n\,.\,n^{k-1}$$

Quam per recursionum identitatem accipitur:

$$S\overset{k}{V}[1..n] = (n + g)^{k+h}$$

pro $k = 1$ adest $S\overset{k}{V}[1..n] = n$ i. e. $g = \delta$ et $h = o$, quare habeatur oportet:

$$S\overset{k}{V}[1..n] = n^k$$

Reperitur inter variationes numeri propositi haec relatio securrens:

$$^{n}\overset{k}{V}[1..] = {}^{n-1}_{1.}\overset{k-1}{V}[1..] + {}^{n-2}_{2.}\overset{k-1}{V}[1..] \ldots$$
$$+ {}^{n-h}_{h.}\overset{k-1}{V}[1..] \ldots + {}^{k-1}_{n-(k-1)}\overset{k-1}{V}[1..]$$

habetur itaque:

$$S^{n}\overset{k}{V}[1..] = S^{n-1}\overset{k-1}{V}[1..] + S^{n-1}\overset{k-1}{V}[1..] \ldots$$
$$+ S^{n-h}\overset{k-1}{V}[1..] \ldots + S^{k-1}\overset{k-1}{V}[1..]$$

sive signis simplicioribus substitutis:

$$^{n}\overset{k}{A} = {}^{n-1}\overset{k-1}{A} + {}^{n-2}\overset{k-1}{A} \ldots + {}^{n-h}\overset{k-1}{A} \ldots + {}^{k-1}\overset{k-1}{A}$$

Adest autem relatio facultatum:

$$\frac{1}{k}.\,{}^{n}\overset{k-1}{\mathfrak{F}}{}^{-1} = {}^{n-1}\overset{k-2}{\mathfrak{F}}{}^{-1} + {}^{n-2}\overset{k-2}{\mathfrak{F}}{}^{-1} \ldots + {}^{n-h}\overset{k-2}{\mathfrak{F}}{}^{-1} ..$$
$$+ {}^{k-1}\overset{k-2}{\mathfrak{F}}{}^{-1}$$

i. e.

$$\frac{-k.}{p}\,{}^{n}\overset{k-1}{\mathfrak{F}}{}^{-1} = \frac{-(k-1)}{p.}\,{}^{n-1}\overset{k-2}{\mathfrak{F}}{}^{-1} + \frac{-(k-1)}{p.}\,{}^{n-2}\overset{k-2}{\mathfrak{F}}{}^{-1} \ldots$$
$$+ \frac{-(k-1)}{p.}\,{}^{n-h}\overset{k-2}{\mathfrak{F}}{}^{-1} \ldots + \frac{-(k-1)}{p.}\,{}^{k-1}\overset{k-2}{\mathfrak{F}}{}^{-1}$$

Quae formula recursionis quia cum illa ipsius ${}^{n}\overset{k}{A}$ est consentiens, concludi potest, esse:

$$S^{n}\overset{k}{V}[1..] = {}^{-(k+h)}_{p.}\;{}^{n+g}\overset{k+h-1}{\mathfrak{F}}{}^{-1}$$

Ad constantes determinandas ponamus $k = 1$, quo facto emergit $h = o$; in expressione ${}^{-k}_{p.}\;{}^{n+g}\overset{k-1}{\mathfrak{F}}{}^{-1}$ ut determinetur constans g, ponatur $n = k$, ex quo fit $S^{n}\overset{k}{V}[1..] = 1$, sequitur ergo, esse $g = -1$ atque generaliter inveniri:

$$S^{n}\overset{k}{V}[1..] = {}^{-k}_{p.}\;{}^{n-1}\overset{k-1}{\mathfrak{F}}{}^{-1} = \frac{(n-1)(n-2)\ldots(n-h)}{k.(k-1)\ldots 2.1}$$

Elementa si ab o incipiant reperitur formula recursionis:

$${}^{n}\overset{k}{V}[o..] = {}^{n}_{o.}\overset{k-1}{V}[o..]\ldots + {}^{n-1}_{1.}\overset{k-1}{V}[o..]\ldots$$
$$+ {}^{n-h}_{n.}\overset{k-1}{V}[o..]\ldots + {}^{o}_{n.}\overset{k-1}{V}[o..]$$

quae habetur ex formula generali, si pro p substituatur valor o.

Adest igitur:

$$S^{n}\overset{k}{V}[o..] = S^{n}\overset{k-1}{V}[o..] + \ldots + S^{n-h}\overset{k-1}{V}[o..]\ldots$$
$$+ S^{o}\overset{k-1}{V}[o..]$$

sive signis acceptis simplicioribus:

$${}^{n}\overset{k}{A} = {}^{n}\overset{k-1}{A} + {}^{n-1}\overset{k-1}{A} \ldots + {}^{n-h}\overset{k-1}{A} \ldots + {}^{0}\overset{k-1}{A}$$

Attamen habetur:

$$\frac{1}{k+1}\cdot {}^{n}\overset{k}{\mathfrak{V}}{}^{1} = {}^{n}\overset{k-1}{\mathfrak{V}}{}^{1} + {}^{n-1}\overset{k-1}{\mathfrak{V}}{}^{1} \ldots + {}^{n-h}\overset{k-1}{\mathfrak{V}}{}^{1} \ldots$$
$$+ {}^{1}\overset{k-1}{\mathfrak{V}}{}^{1} + {}^{0}\overset{k-1}{\mathfrak{V}}{}^{1}$$

sive omnibus terminis numero $\overset{-k}{p}$ multiplicatis:

$$\overset{-(k+1)}{p}\,{}^{n}\overset{k}{\mathfrak{V}}{}^{1} = \overset{-k}{p}\cdot{}^{n}\overset{k-1}{\mathfrak{V}}{}^{1} + \overset{-k}{p}\cdot{}^{1}\overset{k-1}{\mathfrak{V}}{}^{1} \ldots + \overset{-k}{p}\cdot{}^{n-h}\overset{k-1}{\mathfrak{V}}{}^{1} \ldots$$
$$+ \overset{-k}{p}\,{}^{0}\overset{k-1}{\mathfrak{V}}{}^{1}$$

ex quarum identitate colligitur, esse constantibus rite fictis:

$$S^{n}\overset{k}{V}[o..] = \overset{-(k+h+1)}{p}\cdot{}^{n+g}\overset{k+h}{\mathfrak{V}}{}^{1}$$

Pro $n = o$ adest $S^{n}\overset{k}{V}[o..] = 1$ ex quo satis apparet, esse $g = 1$. Pro $k = 2$ adest $S^{n}\overset{k}{V}[o..] = n + 1$, ex quo facile perspicitur, inveniri $h = -2$, sic ergo ut habeatur:

$$S^{n}\overset{k}{V}[o..] = \overset{-(k-1)}{p}\cdot{}^{n+1}\overset{k-2}{\mathfrak{V}}{}^{1} = \frac{(n+1)(n+2)\ldots(n+k-1)}{(k-1)\ldots 2.1}$$

Numerum denique formarum variationis, quae, ad eandem summam, ex elementis $1, 2 \ldots$ omnibus admissis classibus producuntur, ut determinemus, n memoriam revocamus recursionis formulam:

$$ {}^{n}V[1..] = {}^{n-1}_{1.}V[1..] + {}^{n-2}_{2.}V[1..] \ldots $$
$$ + {}^{n-h}_{h.}V[1..] \ldots + {}^{1}_{(n-1)}V[1..] + n $$

adest itaque:

$$ S^{n}V[1..] = S^{n-1}V[1..] \ldots + S^{n-h}V[1..] \ldots $$
$$ + S^{1}V[1..] + 1 $$

recurrit eodem autem modo 2^{n-1} nam:

$$ 2^{n-1} = 2^{(n-1)-1} + 2^{(n-1)-2} \ldots + 2^{(n-1)-h} \ldots $$
$$ + 2^{(n-1(-(n-1)} + 1 $$

habetur ergo:

$$ S^{n}V[1..] = 2^{n-1+g} \ldots $$

pro $n = 1$ adest $S^{n}V[1..] = 1$ ex quo sequitur esse $h = 0$. i. e.

$$ S^{n}V[1..n] = 2^{n-1} $$

Quod autem jam ex relatione colligitur:

$$ S^{n}\overset{k}{V}[1..] = {}^{n-1}\overset{k-1}{\mathfrak{B}}. $$

notum enim est, accipi:

$$ S^{n}V[1..] = S^{n}\overset{1}{V}[1..] + S^{n}\overset{2}{V}[1..] \ldots $$
$$ + S^{n}\overset{k}{V}[1..] \ldots + S^{n}\overset{n}{V}[1..] $$
$$ = 1 + {}^{n-1}\overset{1}{\mathfrak{B}} + {}^{n-1}\overset{2}{\mathfrak{B}} \ldots + {}^{n-1}\overset{k}{\mathfrak{B}} \ldots {}^{n-1}\overset{n-1}{\mathfrak{B}} $$
$$ = 2^{n-1} $$

Theoremate binomiali pro exponentibus integris demonstrando si primitive i. e. non ex theoremate polynomiali, ut postulat scientia, deducatur, hoc argumentum demonstrationem in arctissimum contrahit.

Sit:

$$(1+x)^n = 1 + {}^{n}\overset{1}{A}\, x + {}^{n}\overset{2}{A}\, x^2 \ldots + {}^{n}\overset{h}{A}\, x^h \ldots$$

habetur:

$$(1+x)^{n+1} = \begin{cases} 1 + {}^{n}\overset{1}{A}\, x + {}^{n}\overset{2}{A}\, x^2 \ldots + {}^{h}\overset{h}{A}\, x^h \ldots \\ \qquad 1.\, x + {}^{n}\overset{1}{A}\, x^2 \ldots + {}^{n}\overset{h-1}{A}\, x^h \end{cases}$$

coefficientibus itaque potentiae $n+1^{tae}$ per ${}^{n+1}\overset{1}{A}$, ${}^{n+2}\overset{2}{A}$ ctr. denotandis adest:

$${}^{n+1}\overset{h}{A} = {}^{n}\overset{h-1}{A} + {}^{n}\overset{h}{A}$$

quia autem invenitur:

$${}^{-(h+1)}_{\quad p}\; {}^{n+1}\overset{h}{\mathfrak{V}}{}^{-1} = {}^{-h}_{\;p}\; {}^{n}\overset{h-1}{\mathfrak{V}}{}^{-1} + {}^{-(h+1)}_{\quad p}\; {}^{n}\overset{h}{\mathfrak{V}}{}^{-1}$$

sequitur, constantibus ambabus riti positis esse:

$${}^{n+1}\overset{h}{A} = {}^{-(h+k+1)}_{\quad p}\; {}^{n+g+1}\overset{h+k}{\mathfrak{V}}{}^{-1}$$

ad quas autem determinandes ponatur: $h = 0$, ex quo satis elucet, esse $k = -1$. Constantem autem g ut determinemus, sciri oportet, esse $^{n+1}\overset{1}{A} = m+1$, nam quia:

$$^{m+1}\overset{1}{A} = {}^{n}\overset{1}{A} + 1$$

atque eodem modo numeri naturales recurrunt:

$$(m+1) = (m)+1$$

sequitur, inveniri generaliter:

$$^{m+1}\overset{1}{A} = m+1+k$$

pro $m+1=0$ adest $^{m+1}\overset{1}{A} = 0$ i. e. $k=0$

Sit igitur nunc in expressione:

$$^{n+1}\overset{h}{A} = {}^{n+1+g}\overset{h-1}{\mathfrak{F}}{}^{-1} : \overset{-h}{p}$$

pro $h = 1$ erit:

$$^{n+1+g}\overset{0}{\mathfrak{F}}{}^{-1} = n+1,$$

ex quo perspicuum est, esse $g=0$. i. e.

$$^{n+1}\overset{h}{A} = \overset{-h}{p} \cdot {}^{n+1}\overset{h-1}{\mathfrak{F}}{}^{-1} = \frac{(n+1)\,n\,(n-1)\ldots(n+1-(h-1))}{h\,(h-1)\ldots 1}$$

Demonstrationem theorematis, secundum quod habeatur:

$$(x+\overset{1}{a}).(x+\overset{2}{a})\ldots(x+\overset{n}{a}) = x^n + \overset{1}{C}'\begin{bmatrix}1 & n\\ a\ldots a\end{bmatrix}.x^{n-1}\ldots$$
$$+ \overset{h}{C}'\begin{bmatrix}1 & n\\ a\ldots a\end{bmatrix}x^{n-h} + \overset{n}{C}'\begin{bmatrix}1 & n\\ a\ldots a\end{bmatrix}$$

strenue perducendam submolestam esse, non ignoratur. Ratione autem probandi identitatis duarum recursionum perfacile ac generaliter agitur.

Licet scilicet hanc aequationem ponere fictam:

$$(x+\overset{1}{a}).(x-\overset{2}{a})\dots(x+\overset{n}{a}) = {}^{n}\overset{0}{A}\,x^{n} + {}^{n}\overset{1}{A}\,x^{n-1}_{\dots} + {}^{n}\overset{h}{A}\,x^{n-h}_{\dots} + {}^{n}\overset{n}{A}\,x^{0}$$

factore sequente $(x + \overset{n+1}{a})$ adjecto, atque serie illa per eum multiplicata producitur:

$$(x+\overset{1}{a}).(x+\overset{2}{a})\dots(x+\overset{n+1}{a}) = \left\{\begin{array}{r} {}^{n}\overset{0}{A}\,x^{n+1} + {}^{n}\overset{1}{A}\,x^{n}\dots + {}^{n}\overset{h+1}{A}\,x^{n-h}\dots + {}^{n}\overset{n}{A}\,x \\ + {}^{n}\overset{0}{A}\,\overset{n+1}{.a.}\,x^{n}\dots + {}^{n}\overset{h}{A}\,\overset{n+1}{.a.}\,x^{n-h}\dots \\ \dots + {}^{n}\overset{n}{A}\,\overset{n+1}{a}\,.x^{0} \end{array}\right.$$

i. e. coefficiens $h + 1^{tus}$ ipsius $(x + \overset{1}{a})\dots(x + \overset{n+1}{a})$, quem analogiae ergo per ${}^{n+1}\overset{h+1}{A}$ denotari oportet, reperietur:

$${}^{n+1}\overset{h+1}{A} = {}^{n}\overset{h+1}{A} + {}^{n}\overset{h}{A}\,\overset{n+1}{.a.}$$

Ex scientia autem conjectandi nota est recursio sequens inter classes combinatorias omissis repetitionibus:

$$\overset{h+1}{C'}\begin{bmatrix}1 & n+1\\ a\dots & a\end{bmatrix} = \overset{k+1}{C'}\begin{bmatrix}1 & n\\ a\dots & a\end{bmatrix} + \overset{n+1}{a.}\,\overset{h}{C'}\begin{bmatrix}1 & n\\ a\dots & a\end{bmatrix}$$

qua ex recursionum identitate constantibus rite fictis sequitur, esse

$$ {}^{n+1}\overset{h+1}{A} = \overset{h+1+k}{C'}\left[\begin{smallmatrix}1 & n+1+g\\ a..a\end{smallmatrix}\right] $$

quas constantes ut determinemus, sit $h = -1$ quo facto habetur ${}^{n+1}\overset{o}{A} = 1$, ex quo colligitur, esse $k = 0$, quia $\overset{o}{C'}[1\,..] = 1$ (nihil combinatorium). Pro $h = n$, adest: $\overset{h+1}{C'}\left[\begin{smallmatrix}1 & n+1+g\\ a\,..\,a\end{smallmatrix}\right] = \overset{n+1}{C'}\left[\begin{smallmatrix}1 & n+1\\ a..a\end{smallmatrix}\right]$ i. e. $g = 0$. quare habetur:

$$ {}^{n+1}\overset{h+1}{A} = \overset{h+1}{C'}\left[\begin{smallmatrix}1 & n+1\\ a..a\end{smallmatrix}\right]. $$

Quotus:

$$ \frac{1}{\overset{o}{a}x^{\alpha} + \overset{1}{a}x^{\alpha+\delta} \ldots + \overset{h}{a}x^{\alpha+h\delta} \ldots + \overset{n}{a}x^{\alpha+n\delta}} $$

si sequente modo fingatur:

$$ \overset{o}{A}x^{\beta} + \overset{1}{A}x^{\beta+d} + \ldots + \overset{h}{A}x^{\beta+hd} \ldots $$

ubi autem facile perspici potest, esse $\beta = -\alpha$ et $\gamma = \delta$, in hanc recursionis formulam statim inciditur:

$$ o = \overset{h}{A}\overset{o}{a} + \overset{h-1}{A}\overset{1}{a} \ldots + \overset{h-r}{A}\overset{r}{a} \ldots + \overset{o}{A}\overset{h}{a} $$

i e.

$$ \overset{h}{A} = -\frac{\overset{h-1}{A}\overset{1}{a} + \overset{h-2}{A}\overset{2}{a} \ldots + \overset{h-r}{A}\overset{r}{a} \ldots \overset{o}{A}\overset{h}{a}}{\overset{o}{a}} $$

sive:

$$\overset{h}{A} = \overset{h-1}{A}\left(-\frac{\overset{1}{a}}{a}\right) + \overset{h-2}{A}\left(-\frac{\overset{2}{a}}{a}\right) \dots + \overset{h-r}{A}\left(-\frac{\overset{r}{a}}{a}\right) \dots$$
$$+ \overset{o}{A}\left(-\frac{\overset{h}{a}}{a}\right)$$

qua cum recursione quoque ea pro ${}^{n}V\left[-\frac{\overset{1}{a}}{a}, .\right]$ est consentiens:

$${}^{h}V\left[-\frac{\overset{1}{a}}{a}\dots\right] = {}^{h-1}V\left[-\frac{\overset{1}{a}}{a}\dots\right].\left(\frac{\overset{1}{a}}{a}\right) \dots + {}^{h-r}V\left[-\frac{\overset{1}{a}}{a}.\right]$$
$$\left(-\frac{\overset{r}{a}}{a}\right) \dots + {}^{o}V\left[-\frac{\overset{1}{a}}{a}..\right]\left(-\frac{\overset{n}{a}}{a}\right)$$

ex quo constante determinata esse:

$$\overset{h}{A} = {}^{h}V\left[\frac{\overset{1}{a}}{1a}, -\frac{\overset{2}{a}}{a} \dots\right]$$

concludi potest.

Eadem deinde via probandi perfacile effici potest, ut theorema taylorianum in forma producatur generaliori. Series tayloriana communis valorem evolvit ipsius $\Delta\varphi(x)$ ita, ut secundum potentias ips. Δx fiat evolutio. Generaliter $\frac{d^r y}{\Delta x^r}$ si per $\overset{r}{\mathfrak{D}}$ denotetur, esse:

$$\Delta x = \overset{1}{\mathfrak{D}}\Delta x + \overset{-2}{p}.\overset{2}{\mathfrak{D}}\Delta x^2 \dots + \overset{-k}{p}\overset{k}{\mathfrak{D}}\Delta x^k \dots$$

Supponatur.

Quam seriem autem ratione gradus differentiae faciemus generaliorem. Signo $\overset{k}{\mathfrak{T}}\Delta^m y$ si utamur pro termino kto ipsius $\Delta^m y$, reperitur:

$$\overset{k}{\mathfrak{T}}\Delta^m y = \underset{1..n}{\overset{h}{S}}\left[(-1)^h\,{}^m\overset{h}{\mathfrak{B}}_{.(m-h)}{}^k\right]\overset{k}{\mathfrak{D}}{}^{-k}_{p.}\Delta x^k$$

ubi signi summationis, S, significatio satis nota est.

In qua autem expressione k non minor, quam m esse potest, ita, ut terminus primus ipsius $\Delta^m y$ sit is, in quo $k=m$. Si itaque expressio $\underset{0..n}{\overset{h}{S}}\left[(-1)^h\,{}^m\overset{h}{\mathfrak{B}}_{(m-h)}{}^k\right]$ per ${}^m\overset{k}{\mathfrak{K}}$ indicetur, erit:

$$\Delta^m y = {}^m\overset{m}{\mathfrak{K}}\overset{m}{\mathfrak{D}}{}^{-m}_{p.}\Delta x^m + \dots$$
$$+\,{}^m\overset{m+r}{\mathfrak{K}}\;\overset{m+r}{\mathfrak{D}}{}^{-(m+r)}_{.\,p\,.}\Delta x^{m+r} + \dots$$

seu quia:

$${}^m\overset{m}{\mathfrak{K}} = \underset{1..n.}{\overset{h}{S}}\left[(-1)^h\,{}^m\overset{h}{\mathfrak{B}}_{(m-h)}{}^m\right]$$
$$= \Delta^m (m-h) = \overset{m}{p}$$

habetur:

$$\Delta^m y = \overset{m}{\mathfrak{D}}\Delta x^m + \dots + {}^m\overset{m+r}{\mathfrak{K}}\;\overset{m+r}{\mathfrak{D}}{}^{-(m+r)}_{p.}\Delta x^{m+r} + \dots$$

Ad coefficientium ${}^m\overset{m+1}{\mathfrak{K}}$, ${}^m\overset{m+2}{\mathfrak{K}}$, ctr. computum regulam habemus independentem, sed valde complicatam; disquiremus igitur, an simplicius exhiberi possit.

Invenitur:

$$ {}^{m}\overset{m+r-1}{K} = {}^{m}\overset{0}{\mathfrak{B}}_{m}{}^{m+r-1} - {}^{m}\overset{1}{\mathfrak{B}}_{(m-1)}{}^{m+r-1}\dots(-1)^{k} $$

$$ {}^{m}\overset{h}{\mathfrak{B}}_{(m-h)}{}^{m+r-1}\dots(-1)^{m-1}\,{}^{m}\overset{m-1}{\mathfrak{B}} \quad \text{et} $$

$$ {}^{m-1}\overset{m+r-1}{K} = {}^{m-1}\overset{0}{\mathfrak{B}}_{(m-1)}{}^{m+r-1}\dots(-1)^{h-1}\,{}^{m-1}\overset{h-1}{\mathfrak{B}}_{(m-h)}{}^{m+r-1}\dots $$

$$ (-1)^{m-2}\,{}^{m-1}\overset{m-1}{\mathfrak{B}} $$

quibus additis habetur:

$$ {}^{m}\overset{m+r-1}{K} + {}^{m-1}\overset{m+r-1}{K} = {}^{m-1}\overset{0}{\mathfrak{B}}_{.m}{}^{m+r-1} - \dots $$

$$ + (-1)^{h}\,{}^{m-1}\overset{h}{\mathfrak{B}}_{(m-h)}{}^{m+r-1}\dots + (-1)^{m-1}\,{}^{m-1}\overset{m-1}{\mathfrak{B}} $$

atque:

$$ {}_{m}\left({}^{m}\overset{m+r-1}{K} + {}^{m-1}\overset{m+r-1}{K}\right) = {}^{m}\overset{0}{\mathfrak{B}}_{m}{}^{m+r} - \dots + (-1)^{h}\,{}^{m-1}\overset{h}{\mathfrak{B}}_{(m-h)}{}^{m+r}\dots $$

$$ + (-1)^{m-1}\,{}^{m}\overset{m-1}{\mathfrak{B}} $$

i. e.

$$ {}^{m}\overset{m+r}{K} = m.\left({}^{m}\overset{m+r-1}{K} + {}^{m-1}\overset{m+r-1}{K}\right) $$

vel, ambabus partibus aequationis numero permutationis m^{ti} gradus i. e. per $\overset{m}{p}$ divisis seu per p^{-m} multiplicatis invenitur formula recursionis sequens:

$$ p^{-m}.\,{}^{m}\overset{m+r}{K} = p^{-(m-1)}.\,{}^{m}\overset{m+r-1}{K} + p^{-(m-1)}.\,{}^{m-1}\overset{(m-1)+r}{K} \quad \text{sive:} $$

$$ p^{-m}.\,{}^{m}\overset{m+r}{K} = p^{-m}.\,\overset{m+(r-1)}{K}_{.m} + p^{-(m-1)}.\,{}^{m-1}\overset{(m-1)+r}{K} $$

eodem autem modo etiam $\overset{r}{C}[1..m]$ recurrit;

$$\overset{r}{C}[1..m] = \overset{r-1}{C}[1..m].m + \overset{r}{C}[1..(m-1)].$$

Quarum recursionum identitate colligitur, haberi constantibus determinatis:

$$\overset{-m}{p}.\ {}^{m}\overset{m+r}{K} = \overset{r}{C}[1..m] \text{ et}$$

$${}^{m}\overset{m+r}{K} = \overset{m}{p}.\overset{r}{C}[1..m]$$

Terminus igitur r^{tus} post initialem ipsius $\Delta^m y$ adest:

$$= \overset{m}{p}.\ \overset{-(m+r)}{p}.\ \overset{r}{C}[1..m].\overset{m+r}{\mathfrak{D}}\Delta x^{m+r}$$

habetur ergo:

$$\Delta^m y = \overset{m}{\mathfrak{D}}\Delta x^m + \overset{m}{p}.\overset{-(m+1)}{p}.\overset{1}{C}[1..m]\overset{m+1}{\mathfrak{D}}.\Delta x^{m+1}\dots$$

$$+ \overset{m}{p}.\overset{-(m+r)}{p}.\overset{r}{C}[1..m].\overset{m+r}{\mathfrak{D}}.\Delta x^{m+r}\dots$$

pro $m=1$ theorema taylorianum in simplicissima forma existit, quo enim casu accepto invenitur:

$$\overset{m}{p}.\ \overset{-(m+r)}{p} = \overset{-(r+1)}{p} \text{ et}$$

$$\overset{r}{C}[1..m] = 1.$$

i. e. generaliter:

$$\overset{m}{p}.\overset{-(m+r)}{p}.\overset{r}{C}[1..m].\overset{m+r}{\mathfrak{D}}.\Delta x^{m+r} = \overset{-(r+1)}{p}.\overset{r+1}{\mathfrak{D}}.\Delta x^{r+1}$$

quare habetur:

$$\Delta y = \overset{1}{\mathfrak{D}}\Delta x + \overset{-2}{p}.\overset{2}{\mathfrak{D}}\Delta x^2 \dots \overset{-r}{p}\overset{r}{\mathfrak{D}}\Delta x^r \dots$$

II.

De transitu ex independente determinatione ad recurrentem.

Ex solutione independente in alteram transitus prout postulat disquisitio partim perfacilis, partim autem obstructus est maximis cum difficultatibus. Formula recursionis pro quadam expressione evolvenda postulatur, ut aut progignatur formula recursionis simpliciter, aut in ea conditiones quaedam sint praestandae. Expressio plurium primariarum quantitatum variis recurrere potest viis; recurrendi ratio si praescripta, interdum in maximas inciditur difficultates saepe adhuc invictas. Inter classes combinatorias omissis repetitionibus nonnullae exstant recursionis formulae; talis autem, si deduci possit, qualis sit, ex classibus prioribus successivis proxime altiorem derivare ita, ut in quoque termino eadem sint elementa, fortuna esset decreta scientiae, quae, quod dolendum, ratione laborat non absoluta, dico fortunam algebrae.

Formula autem recursionis simpliciter, i. e. ulla recurrendi rationis conditione omissa semper procreari potest.

Inter facultates e. gr. si formula recursionis exhibenda sit simpliciter, habetur:

$$(k+1)d.\,{}^{n}\overset{k-1}{\mathfrak{F}}{}^{d} = (k+1).d.n.(n+d)\ldots(n+(k-1)d) \quad \text{et}$$

$$\ldots\ {}^{n-d}\overset{k}{\mathfrak{F}}{}^{d} = (n-d).n.(n+d)\ \ldots\ (n+(k-1)d)$$

invenitur ergo:

$$(k+1)d.\,{}^{n}\overset{k-1}{\mathfrak{F}}{}^{d} + {}^{n-d}\overset{k}{\mathfrak{F}}{}^{d} = [(k+1)d+(n-d)].n.(n+d)\ldots(n+(k-1)d)$$

$$= n.(n+d)\ldots(n+(k-1)d)(n+kd)$$

i. e.

$$(k+1)\,d\,{}^{n}\overset{k-1}{\mathfrak{F}}{}^{d} + {}^{n-d}\overset{k}{\mathfrak{F}}{}^{d} = {}^{n}\overset{k}{\mathfrak{F}}{}^{d}$$

Qua recursionis formula mutata ea pro coefficientibus binomialibus atque aliis expressionibus prodit, discerptione tunc varia ratione in totales transformandis.

Lege exhibendi formas combinatorias quasdam independente cognita, facile est, ex ea formulam recursionis deducere ulla autem formationis conditione omissa.

Regula independens procreandi formarum combinatoriarum classem ex elementis 1, 2, . . . n, absque elementorum repetitione est sequens: Forma prima elementa continet ab 1 usque ad eum cum exponente classis convenientem in seriei naturali. Locus in forma aliqua potest augeri, si, aucto ipso, insuper in locis sequentibus elementa poni possunt altiora. Loco alicujus formae serissimo, qui potest augeri, exigue aucto, atque ceteris quam modicissime completis, eo ipso forma illam augendam proxime sequens construitur. Regula autem tam formam omnium primam progignendi quam generaliter ex qualibet proxime sequentem derivandi, cognita, totam classem exhiberi posse manifestum est.

Qua ex regula independente in recurrentem transiemus.

Forma prima elementa continet ab 1 usque ad eum, cujus index cum classis exponente convenit in serie naturali. Elementum primum, 1, si ab illa abscindatur, forma exsistit, quae elementa 2, 3 etr. usque ad eum complectitur, quod indicis ratione cum exponente classis propositae convenit, i. e. forma prima combinatoria ex elementis 1,2, ctr., nihil est aliud, quam forma prima ex elementis 2, 3 ctr. cui elementum primum est praepositum. Regula independente applicanda ex forma prima, 1 2 3, proximae gradus altioris deducentur ita, ut priusquam locus primus augeatur, omnes sequentes completa sint maximis elementis. Ordo igitur primus ipsius classis nihil est aliud, nisi classis unitate minor ex elementis 2, 3 ctr., quibus autem formis elementum primum est praepositum. Si igitur quaeratur $\overset{k}{C}'[1..n]$, ordo primus reperitur $1.\overset{k-1}{C}'[2..n]$. Ordine deinde primo exhibito locus et primus augetur elemento secundo, quo facto sequentes elementis successive altioribus implentur, i. e. forma prima ceterarum est prima classis propositae, quae produci potest, ex elementis datis praeter primum. Qua forma successive tam diu augetur, quamdiu loci elementa altiora accipere possunt, quibus acceptis forma ultima confecta est. Constat ergo classis proposita duabus partibus, quarum alter est: $1.\overset{k-1}{C}'[2..n]$ alter: $\overset{k}{C}'[2..n]$ invenitur. Habetur itaque formula recursionis sequens:

$$\overset{k}{C}'[1..n] = 1.\overset{k-1}{C}'[2..n] + \overset{k}{C}'[2..n],$$

ex qua omnes supra dictae recursionis formulae pro $\overset{k}{C}'[1..n]$ derivari possunt.

Attamen formis quibusdam recurrentis alicujus formulae praescriptis, disquisitio alius est generis.

Casus perfacilis hujus generis peragemus, quo melius difficultates intelligantur, quae in hoc versentur.

Formula recurrens hic adhibenda ea est pro coefficientibus polynomialibus.

Inveniatur formula recursionis inter classes combinatorias admissis repetitionibus ita, ut ex classibus gradus $k-1^{ti}$ ad summas $n-1$, $n-2$, ... $n-r$, .. 1, 0 ex elementis $0, 1, 2$ ... exortis classis producatur k^{ta} summae n^{tae} ex elementis iisdem deducenda, et forma quaelibet multiplicetur in numerum ejus permutationis.

In formula recurrente supra dicta si pro q valor 0 collocatur, habetur relatio:

$$\overset{k}{\underset{p}{{}^{n}C}}\left[\begin{smallmatrix}0\\a..\end{smallmatrix}\right] = \overset{n\ k-1}{\underset{a\ p}{{}^{0}C}}\left[\begin{smallmatrix}0\\a..\end{smallmatrix}\right] + \overset{n-1\ k-1}{\underset{a\ p}{{}^{1}C}}\left[\begin{smallmatrix}0\\a..\end{smallmatrix}\right] + \ldots$$

$$+ \overset{n-h\ k-1}{\underset{a\ p}{{}^{h}C}}\left[\begin{smallmatrix}0\\a..\end{smallmatrix}\right] \ldots + \overset{k-1}{\underset{a\ p}{{}^{n}{}^{0}C}}\left[\begin{smallmatrix}0\\a..\end{smallmatrix}\right]$$

formula recursionis, cujus autem est, omnibus ex classibus gradus $k-1^{ti}$ summarum n, $n-1$, ... 0, classem k^{tam}, quae ad summam n pertinet, derivare. Quae autem recursionis formula ita transmutari potest ut formam accipiat supra postulatam.

Quod autem fieri, si expressio $\overset{k}{\underset{p}{{}^{n}C}}\left[\begin{smallmatrix}0\\a..\end{smallmatrix}\right]$ annihiletur, satis apparet. Ad quod efficiendum hypothesin faciatur, quemque recursionis scalae terminum factorem accipere proprium. Qui autem factor, quia proprius est termini, in quo inest, ab eodem dependet numero, qui terminum indicat scalae recursionis eum, in quo versatur, i. e. factor in k^{to} termino ex h dependit, sive functio est ipsius h. Quam in functionem ipsius h, si pro h valor 1 substituatur, factor primus sive is termini primi recursionis scalae producitur, et ita porro; ex quo intelligi potest, quemque terminum scalae recursionis termino seriei, pro qua functio illa est legis expressio formalis, multiplicari. Casu autem simplicissimo accepto, in quo functio illa est primi gradus, h. e. in quo series illa est series arithmetica primi gradus:

$$b,\ (b+d),\ (b+2d),\ \ldots\ (b+hd),\ \ldots\ (b+nd)$$

erit recursionis scala:

$$b\,.\,{}^{o}_{a}\,.\,{}^{n}_{p}\overset{k-1}{C}\left[{}^{o}_{a..}\right] + (b+d)\,.\,{}^{1}_{a}\,.\,{}^{n-1}_{p}\overset{k-1}{C}\left[{}^{o}_{a..}\right]\ldots$$
$$+\ (b+hd)\,{}^{h}_{a}\,{}^{n-h}_{p}\overset{k-1}{C}\left[{}^{o}_{a..}\right]\ldots + (h+nd)\,{}^{n}_{a}\,{}^{o}_{p}\overset{k-1}{C}\left[{}^{o}_{a..}\right]$$

Expressio autem ${}^{n}_{p}\overset{k}{C}\left[{}^{o}_{a..}\right]$ quomodo hoc ipso mutata est nunc cognoscemus.

Recursionis:

$${}^{n}_{p}\overset{k}{C}\left[{}^{o}_{a..}\right] = {}^{o}_{a}\,{}^{n}_{p}\overset{k-1}{C}\left[{}^{o}_{a..}\right] + \ldots + {}^{h}_{a}\,{}^{n-h}_{p}\overset{k-1}{C}\left[{}^{o}_{a..}\right]\ldots{}^{n}_{a}\,{}^{o}_{p}\overset{k-1}{C}\left[{}^{o}_{a..}\right]$$

ope elementa in complexionibus in eo ordine, qui in arte combinatoria est praescriptus, i. e. secundum quem posteriora elementa nusquam sequuntur priora, non producuntur. Forma combinatoria, M, ipsius ${}^{n}_{p}\overset{k}{C}\left[{}^{o}_{a..}\right]$, cui est numerus permutationis N, ex ${}^{n}_{p}\overset{k}{C}\left[{}^{o}_{a..}\right]$ excepta, quae elementum habeat

k^{tum}, hoc in priori loco poni potest; sequitur ergo partem aliquam ips. N.M ex termino h^{to} scalae recursionis esse exortam. Elementum h^{tum} si in M φ^{ties} versetur, ita ergo, ut sit:

$$ {}^{h,\varphi}S\, h\varphi = n \quad \text{et} \quad S\varphi = k $$

numerum permutationis formae M, qui est in

$\overset{k-1}{{}^{n\,n-h}_{a\;\;\;,p}C}\left[{}^{o}_{a..}\right]$ inveniri:

$$ = \frac{N.\varphi}{k} $$

satis perspicuum est.

Quae autem forma insuper factorem accipit $(b+hd)$ ex quo sequitur, eam partem ipsius NM, quae ex h^{to} termino scalae recursionis trahatur, esse:

$$ = \frac{(b+hd).N.\varphi M}{k} $$

atque ea, quae NM ex omnibus recursionis scalae terminis accipiat, ex hac expressione exsistere, si in ea pro φ et h omnes ponantur valores:

$$ {}^{h,\varphi}S\left[\frac{(b+hd).N.\varphi.M}{k}\right] = \left[\tfrac{b}{k}S\varphi + \tfrac{d}{k}S\,h\varphi\right]N.M $$
$$ = \frac{bk+dn.}{k}\, N.M $$

In qua expressione denique pro M et N si omnes substituantur valores, ea exsistit, in quam $\overset{k}{{}^{n}_{p}C}\left[{}^{o}_{a..}\right]$ sumptione illa est mutata:

$$ {}^{N.M}S\left[\left(\frac{bk+dn}{k}\right)NM\right] = \frac{bk+dn.}{k}\, \overset{k}{{}^{n}_{p.}C}\left[{}^{o}_{a..}\right] $$

Nunc autem quaestio illa redit, quomodo ${}^{n}_{p}\overset{k}{C}\left[{}^{o}_{a..}\right]$ possit annihilari, seu, quomodo b et d sint determinandae, ut fiat:

$$\frac{bk + dn}{k} = o.$$

quod habetur, si $b = -n$ atque $d = k$, erit ergo:

$$o = -n.\,{}^{o\,n}_{a\,p}\overset{k-1}{C}\left[{}^{o}_{a..}\right] + {}_{(-n+k)}\,{}^{1}_{a}\,{}^{n-1}_{p}\overset{k-1}{C}\left[{}^{o}_{a..}\right] \ldots + {}_{(-n+hk)}\,{}^{h}_{a.}\,{}^{n-h}_{p.}\overset{k-1}{C}\left[{}^{o}_{a..}\right] \ldots + {}_{(-n+nk)}\,{}^{n}_{a}\,{}^{o}_{p.}\overset{k-1}{C}\left[{}^{o}_{a..}\right]$$

atque scala recursionis gaudemus sequente, pro $k-1$ valore k substituto:

$$ {}^{p}_{n}\overset{k}{C}\left[{}^{o}_{a..}\right] = \frac{(k+1-n)_{.,}\,{}^{1}_{a}\,{}^{n-1}_{p.}\overset{k}{C}\left[{}^{o}_{a..}\right] \ldots (kh+h-n)_{,}\,{}^{h}_{a}\,{}^{n-h}_{p.}\overset{k-1}{C}\left[{}^{o}_{a..}\right] \ldots k.n\,{}^{n}_{a.}\,{}^{o}_{p}\overset{k-1}{C}\left[{}^{o}_{a..}\right]}{{}^{o}_{n.a}} $$

Quod exemplum ad implicatissima non spectans difficultates disquisitionum hujus generis satis aute oculos ponit.

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